

HIGHER WEIGHTS OF GRASSMANN CODES IN TERMS OF PROPERTIES OF SCHUBERT UNIONS

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ABSTRACT. We describe the higher weights of the Grassmann codes $G(2, m)$ over finite fields \mathbb{F}_q in terms of properties of Schubert unions, and in each case we determine the weight as the minimum of two explicit polynomial expressions in q .

1. INTRODUCTION

Let $G(2, m)$ be the Grassmann variety of 2-dimensional subspaces of a fixed m -dimensional vector space V over the field \mathbb{F}_q with q elements. By the standard Plücker coordinates $G(2, m)$ is embedded into \mathbf{P}^{k-1} as a non-degenerate smooth subvariety, where $k = \binom{m}{2}$. Let $C(2, m)$ be a code with $k \times n$ generator matrix M where the n columns of M viewed as vectors in \mathbb{F}_q^k are the coordinate representatives of the points of $G(2, m)$ under the Plücker embedding. Thus the columns of M are only determined up to a non-zero multiplicative constant, and the ordering of the columns is arbitrary. Nevertheless, the word length n , the dimension k , and the higher weights d_1, \dots, d_k are uniquely determined and independent of the choice of column order and multiplicative constants. It is well known that the word length (the number of \mathbb{F}_q -rational points on $G(2, m)$) is

$$(1) \quad n = \frac{(q^m - 1)(q^{m-1} - 1)}{(q^2 - 1)(q - 1)},$$

and that the dimension is $k = \binom{m}{2}$ as suggested by the choice of notation. In the thesis [P] a formula for the higher weights d_i was given in terms of properties of a certain Young diagram.

Moreover it is well known that for $i = 1, \dots, k$

$$d_i = n - J_i,$$

where J_i is the maximal number of \mathbb{F}_q -rational points from $G(2, m)$ that you can find on a codimension i linear subspace of the Plücker projective space \mathbf{P}^{k-1} . In [HJR] one conjectured that the higher weights were computed by the so-called Schubert unions, in the sense that for any $i \in \{0, 1, 2, \dots, k\}$ the maximal number of \mathbb{F}_q -rational points from $G(2, m)$ that you can find on a codimension i linear subspace of the Plücker space \mathbf{P}^{k-1} can always be found on a codimension i subspace that intersects $G(2, m)$ in such a Schubert union.

In this paper we show that the formula given in [P] is identical with the one predicted by the conjecture concerning Schubert unions, and that Schubert unions thus compute the higher weights. Moreover we may then utilize a procedure (Proposition 4.6 of [HJR], and Proposition 5.3 of [HJR2]) for computing the optimal Schubert unions, in the sense that they contain the maximal number of \mathbb{F}_q -rational points among all Schubert unions spanning

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a linear subspace of Plücker space of the same dimension (at least for all large enough q). In view of the result in [P], it is clear that this procedure, which was described without proof in [HJR] (while a proof was given in [HJR2]), enables us to compute the higher weights of $C(2, m)$ (at least for large q). With permission from the two other authors of [HJR2] we then also give a version of the proof here (with cosmetic changes only). For each m and i this reduces the computation of the higher weight d_i of $C(2, m)$ over \mathbb{F}_q to taking the minimum value of two explicit polynomial expressions in q .

2. BASIC DESCRIPTION OF SCHUBERT UNIONS

In this section we recall the basic facts from [HJR] about Schubert unions, necessary for our purpose. We recall the well known definition of a Schubert variety $\alpha = (a_1, \dots, a_l)$ in the Grassmann variety $G(l, m)$ over a field F , and describe unions of such varieties. Let $B = \{e_1, \dots, e_m\}$ be a basis of a m -dimensional vector space V over F , and let $A_i = \text{Span}\{e_1, \dots, e_i\}$ in V , for $i = 1, \dots, m$. Then $A_1 \subset A_2 \subset \dots \subset A_m = V$ form a complete flag of subspaces of V . With respect to the basis B there is the following canonical cell decomposition of $G(l, m)$.

For a given l -subspace W of V form an $(l \times m)$ -matrix M_W where the rows form a set of basis vectors for W , each row expressed in terms of the basis B .

We choose a basis for W such that the matrix M_W have reduced lower left triangular form, i.e. the last nonzero entry in each row is 1, each of these 1's are the only nonzero entries in their column, and each of these 1's lie in a column to the right of the trailing 1 in the previous row. The trailing 1 in row i is then in column $a_i(W)$ where

$$a_i(W) = \min\{j \in \{1, \dots, l\} : \dim(W \cap A_j) = i\}.$$

For $\alpha = (a_1, \dots, a_l) \in \mathbf{Z}^l$ with $1 \leq a_1 < a_2 < \dots < a_l \leq m$ we define the cell

$$C_\alpha = \{W \in G(l, m) : a_i(W) = a_i \text{ for } i = 1, \dots, l\}.$$

The ordered l -tuples α belong to the grid

$$I(l, m) = \{\beta = (b_1, \dots, b_l) \in \mathbf{Z}^l \mid 1 \leq b_1 < b_2 < \dots < b_l \leq m\}.$$

This grid is partially ordered by $\alpha \leq \beta$ if $a_i \leq b_i$ for $i = 1, \dots, l$.

For each $\alpha \in I(l, m)$ the Schubert variety S_α is defined as:

$$S_\alpha = \{W \in G(l, m) : \dim(W \cap A_{a_i}) \geq i \text{ for } i = 1, \dots, l\} = \bigcup_{\beta \leq \alpha} C_\beta.$$

This is the well-known cell decomposition of S_α .

Next, we choose coordinates for the Plücker space $\mathbf{P}^{k-1} = \mathbf{P}(\wedge^l V)$, with respect to the chosen basis B . Our choice of Plücker coordinates are the maximal minors of the matrix M_W . These minors are indexed as $\{X_\alpha(W) \mid \alpha \in I(l, m)\}$.

Definition 2.1. For any $\alpha \in I(l, m)$, let $I_\alpha(l, m) = \{\beta \in I(l, m) \mid \beta \leq \alpha\}$.

For any $\alpha \in I(l, m)$, it is readily seen that

$$S_\alpha = \{W \in G(l, m) : X_\beta(W) = 0 \text{ for all } \beta \in I(l, m) \setminus I_\alpha(l, m)\}.$$

Also, observe that for $\alpha = (m - l + 1, \dots, m - 1, m)$, we get $S_\alpha = G(l, m)$ and $I_\alpha(l, m) = I(l, m)$.

Definition 2.2. For a subset E of $G(l, m) \subset \mathbf{P}(\wedge^l V)$, let $\mathcal{L}(E)$ be the linear span of E in the projective Plücker space $\mathbf{P}(\wedge^l V)$, and let $L(E)$ the linear span of the affine cone of E in the affine cone of the projective Plücker space $\mathbf{P}(\wedge^l V)$.

We will consider finite intersections and finite unions of such Schubert varieties S_α with respect to our fixed flag. Set $\alpha_i = (a_{i,1}, a_{i,2}, \dots, a_{i,l})$, for $i = 1 \dots, s$. It is clear that: $\cap_{i=1}^s S_{\alpha_i} = S_\gamma$, where $\gamma = (g_1, \dots, g_l)$, and $g_j = \min\{a_{1,j}, \dots, a_{s,j}\}$ for $j = 1, \dots, l$. Thus the intersection of a finite set of Schubert varieties S_α is again a Schubert variety. In particular $\dim L(\cap S_{\alpha_i})$ is equal to the cardinality of $I_\gamma(l, m)$.

Definition 2.3. Given any $U \subset I(l, m)$, the Schubert union S_U and its associated Schubert union grid I_U are defined by

$$S_U = \bigcup_{\alpha \in U} S_\alpha \quad \text{and} \quad I_U = \bigcup_{\alpha \in U} I_\alpha(l, m).$$

We observe: $S_U \subset S_V$ if and only if $I_U \subset I_V$.

0	x	x	x	x	x
0	0	x	x	x	
0	0	x	x		
0	0	0			
0	0				
0					

FIGURE 1. Illustration of I_U , where $U = \{(1, 7), (2, 6), (3, 4)\}$.

In the figure we have drawn the grid $I(2, 7) = I_{(6,7)}(2, 7)$ as the union of $\binom{7}{2}$ squares arranged in a triangle. The bottom square is $(1, 2)$, the top left square is $(1, 7)$, and the top right square is $(6, 7)$. The subset $I_U = I_{(1,7)}(2, 7) \cup I_{(2,6)}(2, 7) \cup I_{(3,4)}(2, 7)$ of $I(2, 7)$ is drawn as the union of the squares containing 0's. For the special case $l = 2$ it is clear that we can draw $I(2, m) = I_{(m-1,m)}(2, m)$ this way, with $\binom{m}{2}$ squares arranged in a triangle, for all $m \geq 2$. Then the bottom square is $(1, 2)$, the top left square is $(1, m)$, and the top right square is $(m-1, m)$.

We then make the following important observation:

Observation 2.4. For $l = 2$, and any $m \geq 2$, draw any grid $I_U \subset I(2, m)$ as a union of squares with zeroes, and $I(2, m) - I_U$ as a union of squares with x's, as in the example. Then the division curve between the squares with 0's and the squares with x's is piecewise linear, and never goes upwards to the right.

Another formulation: The number c_i of squares in I_U from column number i decreases strictly with i until it reaches zero, and in each column of $I(2, m)$, the squares included in I_U are the c_i ones closest to the bottom of that column.

Conversely: If we pick c_1 squares from column 1, \dots , c_t squares from column t , from the bottom for each column, for some $t \leq m-1$, and $c_1 > \dots > c_t$, then we obtain a subset of $I(2, m)$ which is equal to I_U for some Schubert union.

For more details, see [HJR], Section 3. Since each Schubert variety S_α has a decomposition of cells C_β , and all finite intersections of these Schubert varieties are again Schubert varieties, the union S_U also has a cell-decomposition inherited from $G(l, m)$:

$$S_U = \bigcup_{\alpha \in I_U} C_\alpha$$

The following is basically Proposition 2.3 from [HJR].

Proposition 2.5. *Let $U \subset I(l, m)$ and let $S^U = \bigcap_{\alpha \in U} S_\alpha$, and $S_U = \bigcup_{\alpha \in U} S_\alpha$.*

- (1) *The intersection S^U is itself a Schubert variety S_γ with $I_\gamma(l, m) = \bigcap_{\alpha \in U} I_\alpha(l, m)$.*
- (2) *$\mathcal{L}(S_U) \cap G(l, m) = S_U$.*
- (3) *$\dim L(S_U)$ equals the cardinality of I_U .*
- (4) *The number of \mathbb{F}_q -rational points on S_U is $g_U(q) = \sum_{(x_1, \dots, x_l) \in I_U} q^{x_1 + \dots + x_l - l(l+1)/2}$.*

We obtain $g_U(q) = \sum_{(x_1, x_2) \in I_U} q^{x_1 + x_2 - 3}$ for $l = 2$. It is well known that for the Grassmann codes $C(l, m)$ the higher weights d_i satisfy

$$d_i = n - J_i,$$

where J_i is the maximal number of \mathbb{F}_q -rational points from $G(l, m)$ on a codimension i linear space in the Plücker space \mathbf{P}^{k-1} . Hence it gives meaning to say that d_i is "computed by a Schubert union U " if a codimension i space in Plücker space intersects $G(l, m)$ in a Schubert union where the number of \mathbb{F}_q -rational points is J_i . Our goal in the next section is to confirm Conjecture 5.4 of [HJR] in the case $l = 2$. This says:

Conjecture 2.6. *The higher weights of the Grassmann codes $G(l, m)$ are always computed by Schubert unions.*

3. DESCRIPTION OF HIGHER WEIGHTS IN TERMS OF YOUNG DIAGRAMS AND SCHUBERT UNIONS

Let Y_m be a set of boxes arranged in $m - 1$ rows, with $m - i$ boxes in row number i for $i = 1, \dots, m - 1$. One inserts the number $2i + j - 3$ in the j 'th box from the left in the i 'th row, for all i, j in question. We display Y_m in the following way (in the example $m = 9$):

0	1	2	3	4	5	6	7
2	3	4	5	6	7	8	
4	5	6	7	8	9		
6	7	8	9	10			
8	9	10	11				
10	11	12					
12	13						
14							

FIGURE 2. The Young tableau Y_9

A strict subtableau of Y_m is an arrangement of a subset of boxes, where one picks λ_i consecutive boxes from the left end of the i 'th row, for $i = 1, \dots, t$, where $t \leq m - 1$ and $\lambda_1 > \lambda_2 > \dots > \lambda_t$, and keep the numbers in each chosen box. One displays the subtableaux by letting all the chosen boxes stay where they are, and removing all the other boxes. The total number of boxes in the subtableau, viz., $\lambda_1 + \dots + \lambda_t$ is called the *area* of that subtableau. An example with $m = 9$ is shown in the left half of the next figure.

In [P](Theorem 6.18) one gives the following result.

Theorem 3.1. *Fix an integer r , where $1 \leq r \leq k$. Let $\mathcal{F} = \{T_1, T_2, \dots, T_p\}$ be the family of distinct strict subtableaux with area r of Y . Let $a_{d,h}$ denote the number of times that the number d occurs in the h th subtableaux $T_h \in \mathcal{F}$. Associate $\gamma_h = \sum_i a_{d,h} q^i$ to the subtableaux T_h for each h . Then for every r , we have $d_{k-r} = n - \max_h \gamma_h$.*

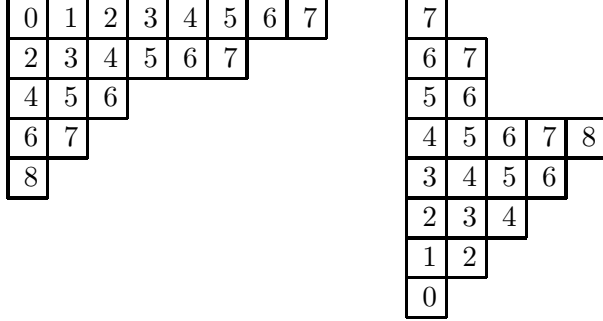


FIGURE 3. A subtableau of Y_9 with $\lambda_1 = 8, \lambda_2 = 6, \lambda_3 = 3, \lambda_4 = 2, \lambda_5 = 1$ and the corresponding Schubert union grid $I_{(1,9)}(2,9) \cup I_{(2,8)}(2,9) \cup I_{(5,6)}(2,9)$

The simple observation is now: There is a bijection between the set of boxes in Y_m and the set of boxes in the grid $I(2, m)$ as follows: The j 'th box from the left in the i 'th row of Y corresponds to the j 'th box from the bottom in the i 'th column of $I(2, m)$, for $i = 1, \dots, m-1$, and all j in question. Then the number inserted in any chosen box B of Y is equal to the Krull dimension of the Schubert variety S_α for the box α_B in $I(2, m)$ that corresponds to B under this bijection. In the right half of the figure above the grid I_U associated to the Schubert union $S_{(1,9)} \cup S_{(2,8)} \cup S_{(5,6)}$ corresponding to $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (8, 6, 3, 2, 1)$ for $m = 9$, is displayed. The Krull dimensions of the S_α for each box α are included.

This immediately gives the crucial:

Lemma 3.2. *There is a one-to-one correspondence between the set of strict subtableaux of Y and the set of subgrids I_U of $I(2, m)$ for all possible Schubert unions S_U (with respect to a fixed chosen flag). Under this correspondence the number γ_h associated to a subtableaux T_h is equal to the number g_U of \mathbb{F}_q -rational points in the Schubert union S_U .*

The results of Theorem 3.1 and Lemma 3.2 taken together confirm Conjecture 2.6 in the case $l = 2$, and we obtain the main result of this paper:

Theorem 3.3. *The higher weights of the Grassmann codes $G(2, m)$ are always computed by Schubert unions.*

4. SCHUBERT UNIONS WITH A MAXIMAL NUMBER OF POINTS

In this section we will show for each $m \geq 2$ how to find the Schubert unions with the largest number of \mathbb{F}_q -rational points among those Schubert unions of fixed spanning dimension $K = \dim L(S_U)$, for each $K = 0, \dots, k$.

Definition 4.1. Fix a dimension $0 \leq K \leq \binom{m}{2}$, and consider the set of Schubert unions $\{S_U\}_K$ in $G(l, m)$ with spanning dimension K .

Then we order the elements in $\{S_U\}_K$ according to the lexicographic order on the polynomials g_U . In other words $S_U > S_V$ if $\deg g_U > \deg g_V$ or $\deg g_U = \deg g_V$, and the coefficient of g_U is larger than that of g_V in the largest degree where the coefficients differ. We call this the order with respect to g_U .

Furthermore we need:

Definition 4.2. Set $\nu_i = 1 + \dots + i$, and $\mu_i = (m-1) + \dots + (m-i)$ for $i = 1, \dots, m-1$ and set $\nu_0 = \mu_0 = 0$. Now for $1 \leq K \leq \binom{m}{2}$ there exists a unique x such that $\mu_x < K \leq \mu_{x+1}$, and a unique z such that $\nu_z < K \leq \nu_{z+1}$; we set $S_L = S_{(x,m)} \cup S_{(x+1,K-\mu_x+x+1)}$ and $S_R = S_{(z,z+1)} \cup S_{(K-\nu_z,z+2)}$.

Proposition 4.3. Fix a dimension $0 \leq K \leq \binom{m}{2}$. Then S_L or S_R is maximal in $\{S_U\}_K$ with respect to the natural lexicographic order on the polynomials g_U . Furthermore, the one(s) that is(are) maximal with respect to g_U , also has(have) the maximum number of points over \mathbb{F}_q for all large enough q .

Remark 4.4. It is clear that for given m and K , we obtain I_U of the described S_L by “filling up as many columns of the $I(2, m)$ -grid as we can from the left and filling in the remaining boxes (if any) from the bottom in the next column”.

Likewise we obtain the I_U of the described S_R by “filling up as many rows of the $I(2, m)$ -grid as we can from the bottom and filling in the remaining boxes (if any) from the left in the next row”.

Proof. Given the spanning dimension K , let $d = d(K)$ be the maximal Krull dimension for the Schubert unions $\{S_U\}_K$. This Krull dimension is the crucial ingredient in our argument, since the Krull dimension is the degree of the polynomial g_U . We will find the maximal polynomial g_U in the lexicographic order. The fact that the union(s) that is(are) maximal with respect to g_U , also has(have) the maximum number of points over \mathbb{F}_q for all large enough q , is obvious. Our argument is visualized by $I(2, m)$, arranged as a set of squares in a triangle as on the next figure. Each point $(a, b) \in I(2, m)$ defines a Schubert variety $S_{(a,b)}$ with Krull dimension $d(a, b) = a + b - 3$. Therefore the Schubert varieties with a fixed Krull dimension lie on the diagonal

$$D_d = \{(x, y) \mid 1 \leq x < y \leq m, \quad x + y - 3 = d\}.$$

Let as above

$$I_{(a,b)}(2, m) = \{(x, y) \in I(2, m) \mid x \leq a, y \leq b\},$$

and

$$I_U = \bigcup_{(a,b) \in U} I_{(a,b)}(2, m).$$

By definition of $d = d(K)$, there is a Schubert union S_U of spanning dimension K with an I_U that contains a point (a, b) on the diagonal D_d , i.e. $a + b - 3 = d$, but there is no such union S_U with I_U -grid that contains a point on the diagonal D_{d+1} .

The cardinality $c(x, y)$ of a $I_{(x,y)}(2, m)$ defines the function

$$c : I(2, m) \rightarrow \mathbb{Z}, \quad (x, y) \mapsto xy - \frac{x(x+1)}{2}.$$

The restriction of this function to the diagonal D_d is defined by

$$c(x, d - x + 3) = x(d - x + 3) - \frac{x(x+1)}{2}, \quad \text{for } \max\{d+2-m, 0\} < x < \frac{d+3}{2}$$

which is clearly quadratic and concave. Therefore it attains its minimum $C(d)$, when x is minimal or maximal, i.e. at one of the end points of the diagonal D_d .

Clearly

$$C(d(K)) \leq K \leq C(d(K) + 1) - 1.$$

We say that a point $(a, b) \in D_{d(K)}$ is **admissible**, if

$$c(a, b) < C(d+1),$$

i.e. has less cardinality than any point in the next diagonal. Equivalently, $(a, b) \in D_d$ is admissible if $I_{(a,b)}(2, m) \subset I_U$ for some Schubert union S_U of spanning dimension at least K . For us the crucial fact is that for $(a, b) \in D_d$, being admissible is a necessary condition for having $I_{(a,b)}(2, m) \subset I_U$ for some Schubert union S_U of spanning dimension exactly K .

Next, we characterize the admissible points by which diagonal D_d they belong to.

Lemma 4.5. *Consider the diagonal*

$$D_d = \{(x, y) | x + y - 3 = d\} = \{(x, d - x + 3) \mid \max\{d + 2 - m, 0\} < x < \frac{d + 3}{2}\}$$

(i) *Let $d \leq m - 3$, then the only admissible point on D_d is $(1, d + 2)$, except when $d = 2$, where $(2, 3)$ is also admissible.*

(ii) *Let $d > m - 3$, then $(x, d - x + 3)$ is an admissible point on the diagonal D_d only if $d + 3 - m \leq x \leq d + 4 - m$ or $\frac{d}{2} \leq x < \frac{d+2}{2}$, with one exception, namely when $m = 11$ and $d = 10$, then the point $(4, 9)$ is also admissible.*

(iii) *If $m > 10$, then the point (x, m) is admissible, only if $x \geq m - 3$ or $x \leq \frac{m}{5} + 2$ if $x + m$ is odd, and only if $x \geq m - 3$ or $x \leq \frac{m}{5} + 1$ if $x + m$ is even.*

(iv) *If $m > 10$, then the point $(x, m - 1)$ is admissible only if $x \geq m - 4$ or $x \leq \frac{m}{5} + 1$ if $x + m$ is odd, and only if $x \geq m - 4$ or $x \leq \frac{m}{5} + 2$ if $x + m$ is even.*

Remark 4.6. (a) The lemma implies, in very rough terms, that apart from the leftmost column, the two uppermost rows, and the two right-lowest points on each diagonal (it's really only necessary to consider the right half of them, in addition to $(2, 3)$) then the "interior" that remains contains no admissible points (except $(4, 9)$ for $m = 11$).

(b) If d is odd, then part (ii) of the lemma implies that the point of the diagonal D_d with the next to largest x , is non-admissible, so that the admissible ones must be among the two to the left, and the one to the right.

As an illustration we now show the grid $I(2, 15)$ with the cost values included for each element in the grid:

14	27	39	50	60	69	77	84	90	95	99	102	104	105
13	25	36	46	55	63	70	76	81	85	88	90	91	
12	23	33	42	50	57	63	68	72	75	77	78		
11	21	30	38	45	51	56	60	63	65	66			
10	19	27	34	40	45	49	52	54	55				
9	17	24	30	35	39	42	44	45					
8	15	21	26	30	33	35	36						
7	13	18	22	25	27	28							
6	11	15	18	20	21								
5	9	12	14	15									
4	7	9	10										
3	5	6											
2	3												
1													

FIGURE 4. The grid $I(2, 15)$ with the cost values included for each element

As an additional illustration we show the same grid with the non-admissible points crossed out:

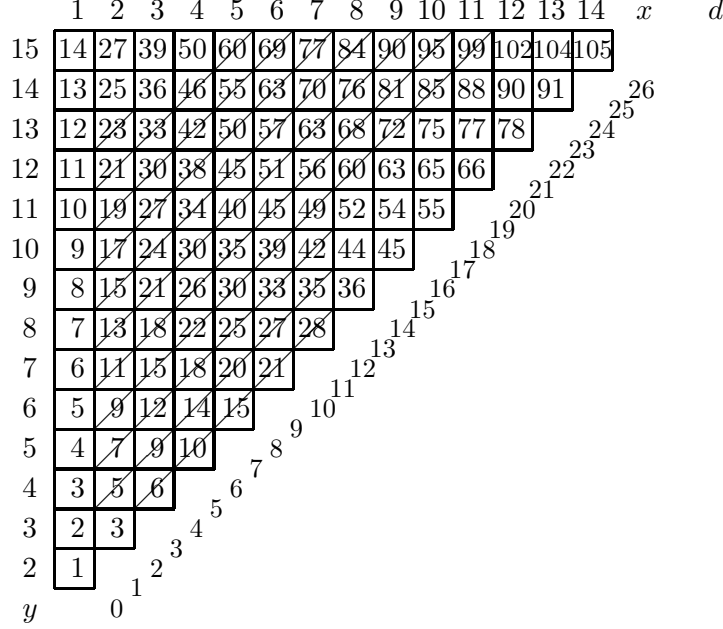


FIGURE 5. $I(2, 15)$ with non-admissible points crossed out

Proof. (i) The cases $d = 0, 1, 2$ are left to the reader. For $d \geq 3$ the concavity of the cardinality function $c(x, y)$ along the diagonals implies that it is enough to check the admissibility of the points $(2, d+1)$, and in addition $(\frac{d+2}{2}, \frac{d+4}{2})$ when d is even, and $(\frac{d+3}{2}, \frac{d+5}{2})$ when d is odd. We compare their c -values with that of $(1, d+1)$ on the diagonal above. We obtain:

$$c(2, d+1) = 2d - 1 > d + 1 = c(1, d+3).$$

Furthermore $c(\frac{d+2}{2}, \frac{d+4}{2}) = \frac{d^2+6d+8}{8} > d + 1 = c(1, d+3)$ when d is even, and $c(\frac{d+3}{2}, \frac{d+5}{2}) = \frac{d^2+8d+7}{8} > d + 1 = c(1, d+3)$ when d is odd.

(ii) Again, by the concavity of the function $c(x, y)$ restricted to the diagonal D_d , it suffices to compare a few points on each diagonal with the end points on the next. Let us first consider the case d even. There is only something to prove if there are at least 5 points on D_d . The number of points on D_d is $\frac{d+2}{2} - (d - m + 3) + 1 = \frac{-d}{2} + m - 1$. This is at least 5 if and only if $d \leq 2m - 12$ and $m \geq 9$. So we assume that. To show the assertion we must show that for each of the values $c(d - m + 5, m - 2)$ and $c(\frac{d-2}{2}, \frac{d+8}{2})$ (for the points third from the left and third from the right on D_d), they cannot be smaller than both of the values $c(d - m + 4, m)$ and $c(\frac{d+2}{2}, \frac{d+6}{2})$ for the endpoints of the diagonal D_{d+1} .

For the point $(d - m + 5, m - 2)$ we obtain two inequalities that reduce to $4m - 15 < 3d$, and

$$(2m - 10 - d)(5d - 6m + 20) < 16.$$

To satisfy this latter inequality, either the factors have different sign or both factors have small positive value. Furthermore we know that $d \leq 2m - 12$, so $2m - 10 - d > 0$, so only the second factor can be negative. In this case, $4m - 15 < 3d$ and $5d - 6m + 20 < 0$, which gives $m \leq 7$, contrary to $m \geq 9$. Thus both factors must have small (even) value. If the first factor $2m - 10 - d$ is 2, i.e. $d = 2m - 12$ we must have $5d - 6m + 20 \leq 6$,

which gives $m \leq 11$ (and $m \geq 9$). A quick check gives $(4, 9)$ as the only admissible point in this case. A further check of the other cases $2m - 10 - d = 4$, and $2m - 10 - d = 6$, only possible for $m \leq 13$ and $m \leq 15$ reveals that the points $(d - m + 5, m - 2)$ in question are non-admissible in these cases.

The point $(\frac{d-2}{2}, \frac{d+8}{2})$ is admissible only if

$$c(\frac{d-2}{2}, \frac{d+8}{2}) < c(d-m+4, m) \quad \text{and} \quad c(\frac{d-2}{2}, \frac{d+8}{2}) < c(\frac{d+2}{2}, \frac{d+6}{2}).$$

The latter inequality yields $d < 12$, and hence $m < d + 3 = 15$, in which case the first inequality is satisfied only if $m = 11$ and $d = 10$, when the point $(4, 9)$ is also admissible.

If d is odd, we compare the value of c at the endpoints $(d-m+4, m)$ and $(\frac{d+3}{2}, \frac{d+5}{2})$ on D_{d+1} , with its value at $(d-m+5, m-2)$ and $(\frac{d-1}{2}, \frac{d+7}{2})$, namely the points number three and two from the endpoints on D_d . We only have an issue if we have at least 4 elements on the diagonal D_d , which gives $d \leq 2m - 11$, and $m \geq 8$.

The point $(d-m+5, m-2)$ is admissible only if

$$c(d-m+5, m-2) < c(d-m+4, m) \quad \text{and} \quad c(d-m+5, m-2) < c(\frac{d+3}{2}, \frac{d+5}{2}).$$

The first inequality reduces to $d > \frac{4m-15}{3}$, while the second one yields

$$5d^2 + (68 - 16m)d + 12m^2 - 100m + 215 > 0.$$

For each fixed m this last inequality has a solution only for d outside an interval $[d_1, d_2]$, where $d_1 < \frac{4m-15}{3}$ and $d_2 > 2m - 11$, at least for $m \geq 11$. For lower m one checks the statement case by case.

The point $(\frac{d-1}{2}, \frac{d+7}{2})$ is admissible only if $c(\frac{d-1}{2}, \frac{d+7}{2}) < c(\frac{d+3}{2}, \frac{d+5}{2})$, which gives $d < 7$, and $m < d - 3$, which is impossible for $m \geq 8$.

(iii) The point (x, m) with $x < m$ is the endpoint with minimal x of the diagonal D_d with $d = x + m - 3$. It has cardinality $xm - \frac{x(x+1)}{2}$, and for $x < m - 3$ it is admissible only if the cardinality of the rightmost endpoint of the diagonal D_{x+m-2} is strictly bigger.

If $x + m$ is odd, then the other endpoint of D_{x+m-2} is $(\frac{x+m-1}{2}, \frac{x+m+3}{2})$ and the inequality becomes

$$xm - \frac{x(x+1)}{2} < \frac{x+m-1}{2} \frac{x+m+3}{2} - \frac{1}{2} \frac{x+m-1}{2} \frac{x+m+1}{2}$$

which means

$$5x^2 + (8 - 6m)x + (m^2 + 4m - 5) > 0$$

For each fixed m this last inequality has a solution only for x outside an interval $[x_1, x_2]$, where $x_1 < \frac{m}{5} + 3$, and $x_2 > m - 4$. The value $x = \frac{m}{5} + 2$ satisfies the inequality only if $m \leq 13$, and for these low values of m we see that the result holds.

If $x + m$ is even, then this endpoint is $(\frac{x+m}{2}, \frac{x+m+2}{2})$ and the condition for admissibility is

$$xm - \frac{x(x+1)}{2} < \frac{x+m}{2} \frac{x+m+2}{2} - \frac{1}{2} \frac{x+m}{2} \frac{x+m+2}{2} = \frac{1}{8}(x+m)(x+m+2)$$

which gives:

$$5x^2 + (6 - 6m)x + (m^2 + 2m) > 0.$$

For each fixed m this last inequality has a solution only for x outside an interval $[x_1, x_2]$, where $x_1 < \frac{m}{5} + 2$, and $x_2 > m - 3$. The value $x = \frac{m}{5} + 1$ satisfies the inequality only if $m \leq 13$, and for these low values of m we see that the result holds.

(iv) In this case the computation is similar. The point $(x, m-1)$ in the next to upper-row of D_d has cardinality $c((x, m-1)) = x(m-1) - \frac{x(x+1)}{2}$ and is admissible only if it has lower cardinality than the lower endpoint of the diagonal D_{d+1} above. If $x+m$ is odd, then the lower end of the diagonal above is $(\frac{x+m-1}{2}, \frac{x+m+1}{2})$, and its cardinality is $\frac{(x+m+1)(x+m-1)}{8}$. Now the condition

$$x(m-1) - \frac{x(x+1)}{2} < \frac{(x+m+1)(x+m-1)}{8}$$

translates to:

$$(x-m+3)(5x-m-3) + 8 > 0.$$

The first factor is negative unless $m-3 \leq x \leq m-1$, while the second factor is negative when $x < \frac{1}{5}(m+3)$. If we set $x = m-5$, the inequality is satisfied only if $m < 8$. Likewise, if $x = \frac{m}{5} + 1$ the inequality is satisfied only if $m < 10$. By the concavity argument the result follows.

If $x+m$ is even, then the lower end of the diagonal above is $(\frac{x+m-2}{2}, \frac{x+m+2}{2})$, and its cardinality is $\frac{(x+m-2)(x+m+4)}{8}$. The necessary condition for $(x, m-1)$ being admissible is

$$x(m-1) - \frac{x(x+1)}{2} < \frac{(x+m-2)(x+m+4)}{8}.$$

This becomes

$$(x-m+4)(5x-m-6) + 16 > 0.$$

We insert $m-6$ which is the largest x -value smaller than $m-4$ making $x+m$ even and obtain $m \leq 10$. Likewise, we insert $x = \frac{m}{5} + 2$ and obtain that the inequality then holds for $m \leq 12$. Hence the statement of the lemma holds for $m \geq 13$. A special check reveals that it holds for $m = 11, 12$ also. \square

We return to the proof of Proposition 4.3 and assume that S_U is a Schubert union with spanning dimension K , and that S_U has the maximal number of points among such unions, i.e. g_U is maximal in the lexicographical order. Therefore the grid I_U contains an admissible point $\alpha = (x, y)$ in the $d(K)$ -diagonal, i.e. $x+y-3 = d = d(K)$. By Lemma 4.5 it suffices to study the following eight cases.

- (a) $d \leq m-3$
- (b) $m > 10$ and $\alpha = (d-m+3, m)$ with $2 \leq d-m+3 \leq \frac{m}{5}+2$, i.e. $m-1 \leq d \leq \frac{6m}{5}-1$.
- (c) $m > 10$ and $\alpha = (d-m+4, m-1)$, with $2 \leq d-m+4 \leq \frac{m}{5}+2$, i.e. $m-2 \leq d \leq \frac{6m}{5}-2$.
- (d) d is even and $\alpha = (\frac{d+2}{2}, \frac{d+4}{2})$.
- (e) d is odd and $\alpha = (\frac{d+1}{2}, \frac{d+5}{2})$.
- (f) d is even and $\alpha = (\frac{d}{2}, \frac{d+6}{2})$.
- (g) $m = 11$, and I_U intersects the $d(K)$ -diagonal in $(4, 9)$.
- (h) $m \leq 10$.

In each case we consider the residual grid $\Delta = I_U \setminus I_\alpha(2, m)$, and find the diagonal $D_{d'}$ with largest d' that Δ intersects. By the lexicographical ordering of g_U , the value of d' is determined in a similar fashion as $d = d(K)$ by the cardinality of Δ and the shape of the grid $I(2, m) \setminus I_\alpha(2, m)$. Notice that S_U is a finite union of irreducible components, all of them Schubert varieties. Furthermore, the point α corresponds to a Schubert variety component S_α of maximal Krull dimension in S_U , and any point $\beta \in \Delta \cap D_{d'}$ corresponds to a Schubert variety S_β of maximal Krull-dimension among the rest of the irreducible components of S_U . First of all the cardinality of Δ is $e = K - c(\alpha)$, and by definition of $d = d(K)$, the Krull dimension d' of S_β is at most $d(K)$.

Now we treat the 8 cases separately: In case (a), when $d \leq m - 3$, then $\alpha = (2, 3)$ or $(1, d + 2)$. If $\alpha = (2, 3)$, then $d(K) = 2$, so $K = 3$ and $e = K - c(\alpha) = 0$, and $S_U = S_\alpha$, which is a Schubert union of type S_R . Consider the case $\alpha = (1, d + 2)$. If $K \geq d + 2$, then $d(K) > d + 1 = c(\alpha)$, contrary to the assumption, so $K = c(\alpha) = d + 1$. In particular $e = K - c(\alpha) = 0$ and $S_U = S_\alpha$, which is of type S_L .

In case (b), the grid $I(2, m) \setminus I_\alpha(2, m) = \{(x, y) \in I(2, m) | d - m + 4 \leq x < y \leq m\}$. Since $d(K) = d$, the cardinality $e = K - c(\alpha)$ of Δ is less than the cardinality of the leftmost column of $I(2, m) \setminus I_\alpha(2, m)$, i.e. at most $m - (d - m + 5) = 2m - d - 5$. We use an argument of (a), almost identical to that in the $\alpha = (1, d + 2)$ case of (a), to conclude that $\Delta = \{(d - m + 4, y) | d - m + 4 < y \leq e\}$. Notice furthermore that S_U clearly is of type S_L .

In case (c), with $\alpha = (d - m + 4, m - 1)$. Since $d(K) = d$, we first see that the cardinality of Δ is less than the cardinality of the upper row of $I_\alpha(2, m)$, i.e. $e = K - c(\alpha) < d - m + 4$. Compare now the row of points $R = \{(x, m) | 1 \leq x < d - m + 4\}$ with the column $C = \{(d - m + 5, y) | d - m + 5 < y < m\}$, both in $I(2, m) \setminus I_\alpha(2, m)$. Notice that both have cardinality at least e , so that for Δ to reach the maximal diagonal $D_{d'}$, it must be contained in one of these. The row R starts on the diagonal D_{m-2} , while the columns C starts on the diagonal $D_{2d-2m+8}$. When $m > 10$ and $d \leq \frac{6m}{5} - 2$, the highest of these diagonals is D_{m-2} , since then $2d - 2m + 8 \leq \frac{2m}{5} + 4 < m - 2$, so in that case Δ must be completely contained in the row R .

To see whether S_U is of type S_L , there are essentially two different situations: $e = d - m + 3$ (maximum possible), and $e \leq d - m + 2$. If $e = d - m + 3$, we are already in case b), since $(d - m + 3, m)$ and $(d - m + 4, m - 1)$ are on the same diagonal, and the point $(d - m + 3, m)$ is covered by case (b).

Assume $e \leq d - m + 2$. Then S_U is not of type S_L , but we revise I_U , and collectively remove the $d - m + 3 - e$ top squares of the right column of $I_\alpha(2, m)$, and reinstall them horizontally as points $(e + 1, m), (e + 2, m), \dots, (d - m + 3, m)$. This amounts to moving squares along $d - m + 3 - e$ diagonals, and does not alter the number of \mathbb{F}_q -rational points of the Schubert unions represented by the two grids. But after moving, we have the grid of type S_L , as in case (b) and we are done. (We have $d - m + 3$ columns to the left filled up completely, and $K - c(d - m + 3, m)$ squares in column nr $d - m + 4$).

In case (d), with $\alpha = (\frac{d+2}{2}, \frac{d+4}{2})$ the lowest row of $I(2, m) \setminus I_\alpha(2, m)$, if any, has larger length than d' , since $d(K) > d'$. Therefore Δ is completely contained in this lowest row. Furthermore, S_U is clearly of type S_R .

In case (e) we see that $K - c(\alpha) = 0$, since if Δ is non-empty, then it could lie to the right of α in the top row, and $d(K)$ would have been larger. On the other hand $S_U = S_\alpha$ is clearly of type S_R .

In case (f), we see that $K - C(d(K)) = 0$ or 1 , since if it was at least 2 , Δ could lie to the right of α , and then $d(K)$ would have been larger. If Δ is empty, there is nothing to do, and if Δ consists of one point, the choice with the largest g_U is $\Delta = \{(\frac{d+2}{2}, \frac{d+4}{2})\}$ (lies to the right of and below α). In both cases S_U is of type S_R .

Starting with (g), we have $m = 11$, and $\alpha = (4, 9)$. Since α lies on the diagonal D_{10} and $c(4, 9) + 1 = 27 = C(11)$, we see that this is only an issue when $K = c(4, 9) = 26$, i.e. Δ is empty. But a quick calculation reveals that $S_{(4,9)}$ is not optimal at all for $K = 26$ with respect to g_U . An optimal choice in this case is $S_U = S_{(2,11)} \cup S_{(3,10)}$, which is of type S_L . Hence the assertion holds in all cases.

The cases $m \leq 10$ are entirely similar. The check that no case occurs that is not of one of the kinds above is left to the reader. \square

It seems obvious that for each given K and m the one(s) among S_L and S_R which is maximal with respect to the lexicographical order on g_U , has a maximal number of \mathbb{F}_q -rational points, not only for large enough q , but for all fixed prime powers $q \geq 2$. All examples indicate that, but we have not been able to prove it.

Remark 4.7. We see that Proposition 4.3 implies the result hoped for in Remark 29 of [GPP], at least for large enough q . This result says that for the Grassmann code $G(2, m)$, with $m > 4$ we have

$$d_r = (q^\delta + q^{\delta-1} + \cdots + q^{\delta-m+2}) + (q^{\delta-2} + q^{\delta-3} + \cdots + q^{\delta-r+m})$$

and

$$d_{k-r} = n - (1 + q + \cdots + q^{m-2}) - (q^2 + q^3 + \cdots + q^{r-m+2}).$$

for $m < r \leq 2m - 5$.

This follows from the fact that for $m < n - K \leq 2m - 5$ the S_R are maximal with respect to the lexicographical order on the g_U , among those Schubert unions with spanning dimension K , and for $m < K \leq 2m - 5$ the S_L are maximal with respect to the lexicographical order on the g_U , among those Schubert unions with spanning dimension K . One then simply counts the number of \mathbb{F}_q -rational points on these particular unions S_L and S_R .

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